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Sufficiency and duality for multiobjective control problems under generalized (B, ρ)-type I functions

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Abstract In this paper, we introduce the classes of (B, ρ)-type I and generalized (B, ρ)-type I, and derive various sufficient optimality conditions and mixed type duality results for multiobjective control problems under (B, ρ)-type I and generalized (B, ρ)-type I assumptions.

Keywords Multiobjective control problems · Efficient solution · Proper efficient solution · Generalized (B, ρ)-type I functions · Sufficient conditions · Mixed type duality

1 Introduction

There is, by now, an extensive literature on optimal control problems, for example Mond and Smart [10], Bhatia and Kumar [1], Nahak and Nanda [12], Preda [14], Zhian and Qingkai [15], Chen [4], Gulati et al. [7], Patel [13], Nahak [11], and De Oliveira et al. [5].

An important direction of work has been to find duality theorems, necessary and sufficient conditions for optimality using more and more general classes of functions.

In [10] Mond and Smart gave duality results and sufficiency conditions under invexity assumptions. Preda [14] has generalized the results of Mond and Smart under generalized

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ρ-invexity for scalar control problems. In [6], Gramatovici has studied multiobjective control problems under generalized invexity assumptions and extended the results in Preda [14].

In this paper, we will introduce the classes of (B, ρ)-type I functions and generalized (B, ρ)-type I functions and derive a series of sufficient optimality conditions and mixed type duality results for multiobjective control problems.

The control problem is to choose, under given conditions, a control $u(t)$, such that the state vector $x(t)$ is brought from specified initial state $x(a) = \alpha$ to some specified final state $x(b) = \beta$ in such a way to minimize a given functional. A more precise mathematical formulation is given in the following problem

$$(MCP) \begin{cases} \text{Minimize } \int_a^b f(t, x, u)dt = \left(\int_a^b f_1(t, x, u)dt, \ldots, \int_a^b f_p(t, x, u)dt\right) \\ \text{subject to} \\ x(a) = \alpha, \quad x(b) = \beta, \\ h(t, x, u) = \dot{x}, \quad t \in I, \\ g(t, x, u) \leqq 0, \quad t \in I, \end{cases}$$

where $I = [a, b]$ is a real interval, $f : I \times I\!R^n \times I\!R^n \longrightarrow I\!R^p$, $g : I \times I\!R^n \times I\!R^n \longrightarrow I\!R^m$ and $h : I \times I\!R^n \times I\!R^n \longrightarrow I\!R^n$ are continuously differentiable functions with respect to each of their arguments, up to the second order.

2 Preliminaries

Let $I\!R^n$ be an n-dimensional Euclidean space, and $I\!R^n_+$ be its nonnegative orthant.
Let x and y be in $I\!R^n$, we denote

$$\begin{aligned} x \leqq y &\Longleftrightarrow x_i \leq y_i, \quad \text{for } i = 1, \ldots, n. \\ x \leq y &\Longleftrightarrow x \leqq y, \quad \text{but } x \neq y. \\ x < y &\Longleftrightarrow x_i < y_i, \quad \text{for } i = 1, \ldots, n. \end{aligned}$$

For the function $f(t, x, u)$, where $x : I \longrightarrow I\!R^n$ is differentiable with its derivative $\dot{x}$ and $u : I \longrightarrow I\!R^m$ is a smooth function, denote the $p \times n$ and $p \times m$ matrices of first partial derivatives of f with respect to x, u by f_x and f_u, such that

$$f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n}\right) \text{ and } f_{iu} = \left(\frac{\partial f_i}{\partial u_1}, \ldots, \frac{\partial f_i}{\partial u_m}\right), \quad i = 1, 2, \ldots, p.$$

Similarly, g_x and g_u denote the $m \times n$ matrices of first partial derivatives of g with respect to x and u.
Here $x(t)$ is the state variable and $u(t)$ is the control variable, u is related to x via the state equation $h(t, x, u) = \dot{x}$.
Let $\mathcal{C}(I, I\!R^n)$ denote the space of piecewise smooth state functions x with norm

$$\|x\| := \|x\|_\infty + \|Dx\|_\infty,$$

where the differential operator D is given by

$$y = Dx \Longleftrightarrow x(t) = x(a) + \int_a^t y(s)ds.$$

Therefore, $D = d/dt$ except at discontinuities.
Let X be the space of continuously differentiable state functions $x : I \to I\!R^n$ such that $x(a) = \alpha$, $x(b) = \beta$, equipped with the norm $\|x\| := \|x\|_\infty + \|Dx\|_\infty$ and U is the space of piecewise continuous control functions $u : I \to I\!R^m$ has the uniform norm $\|\cdot\|_\infty$.

Definition 1 A point $(x^*, u^*) \in X \times U$ is said to be an efficient (Pareto optimal) solution of (MCP) if, there exists no other $(x, u) \in X \times U$ such that

$$\int_a^b f(t, x, u)dt \leq \int_a^b f(t, x^*, u^*)dt.$$

Definition 2 An efficient solution (x^*, u^*) of (MCP) is said to be properly efficient [2] if there exists a positive number M such that for each i, we have

$$\int_a^b f_i(t, x^*, u^*)dt - \int_a^b f_i(t, x, u)dt \leqq M\left(\int_a^b f_j(t, x, u)dt - \int_a^b f_j(t, x^*, u^*)dt\right),$$

for some j such that $\int_a^b f_j(t, x, u)dt > \int_a^b f_j(t, x^*, u^*)dt$ whenever $(x, u) \in X \times U$ and $\int_a^b f_i(t, x, u)dt < \int_a^b f_i(t, x^*, u^*)dt$.

Let $\rho = (\rho^1, \ldots, \rho^p)$ be a vector in $I\!R^p$ and $d : I \times I\!R^n \times I\!R^n \longrightarrow I\!R$ be a piecewise continuous function.

Definition 3 A pair (ϕ, ψ) is said to be (B, ρ)-type I at $(x^*, u^*) \in \mathcal{C}(I, I\!R^n) \times U$ with respect to b_0, b_1, η, ξ if there exist functions $b_0(x, x^*, u, u^*) \in I\!R_+$, $b_1(x, x^*, u, u^*) \in I\!R_+$, $\eta(t, x(t), x^*(t)) \in I\!R^n$ and $\xi(t, u(t), u^*(t)) \in I\!R^m$, with $\eta(t, x(t), x^*(t)) = 0$ if $x(t) = x^*(t)$, such that for all $(x, u) \in X \times U$,

$$\begin{aligned} & b_0(x, x^*, u, u^*)\left[\int_a^b \phi(t, x, \dot{x}, u)dt - \int_a^b \phi(t, x^*, \dot{x}^*, u^*)dt\right] \\ & \geqq \int_a^b \left[\eta^t \phi_x(t, x^*, \dot{x}^*, u^*) + [D\eta]^t \phi_{\dot{x}}(t, x^*, \dot{x}^* u^*)\right. \\ & \left. + \xi^t \phi_u(t, x^*, \dot{x}^*, u^*)\right] dt + \rho^1 \int_a^b d^2(t, x, x^*)dt, \qquad (1) \\ & - b_1(x, x^*, u, u^*) \int_a^b \psi(t, x^*, \dot{x}^*, u^*)dt \\ & \geqq \int_a^b \left[\eta^t \psi_x(t, x^*, \dot{x}^*, u^*) + [D\eta]^t \psi_{\dot{x}}(t, x^*, \dot{x}^*, u^*)\right. \\ & \left. + \xi^t \psi_u(t, x^*, \dot{x}^*, u^*)\right] dt + \rho^2 \int_a^b d^2(t, x, x^*)dt. \end{aligned}$$

If in the previous definition, (1) is satisfied as a strict inequality, then we say that a pair (ϕ, ψ) is semistrictly (B, ρ)-type I at $(x^*, u^*) \in \mathcal{C}(I, I\!R^n) \times U$ with respect to b_0, b_1, η and ξ.

Now we generalize the class of (B, ρ)-type I functions to a class of functions called generalized (B, ρ)-type I.

Definition 4 A pair (ϕ, ψ) is said to be weak strictly (B, ρ)-pseudo-quasi-type I at $(x^*, u^*) \in \mathcal{C}(I, I\!R^n) \times U$ with respect to b_0, b_1, η, ξ if there exist functions $b_0(x, x^*, u, u^*) \in I\!R_+$, $b_1(x, x^*, u, u^*) \in I\!R_+$, $\eta(t, x(t), x^*(t)) \in I\!R^n$ and $\xi(t, u(t), u^*(t)) \in I\!R^m$, with $\eta(t, x(t), x^*(t)) = 0$ if $x(t) = x^*(t)$, such that for all $(x, u) \in X \times U$,

$$\int_a^b \phi(t, x, \dot{x}, u)dt \leq \int_a^b \phi(t, x^*, \dot{x}^*, u^*)dt$$

$$\Longrightarrow b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t \phi_x(t, x^*, \dot{x}^*, u^*) + [D\eta]^t \phi_{\dot{x}}(t, x^*, \dot{x}^* u^*)\right.$$

$$\left. + \xi^t \phi_u(t, x^*, \dot{x}^*, u^*)\right] dt < -\rho^1 \int_a^b d^2(t, x, x^*)dt,$$

$$- \int_a^b \psi(t, x^*, \dot{x}^*, u^*)dt \leqq 0$$

$$\Longrightarrow b_1(x, x^*, u, u^*) \int_a^b \left[\eta^t \psi_x(t, x^*, \dot{x}^*, u^*) + [D\eta]^t \psi_{\dot{x}}(t, x^*, \dot{x}^*, u^*)\right.$$

$$\left. + \xi^t \psi_u(t, x^*, \dot{x}^*, u^*)\right] dt \leqq -\rho^2 \int_a^b d^2(t, x, x^*)dt.$$

Definition 5 A pair (ϕ, ψ) is said to be strong (B, ρ)-pseudo-quasi-type I at $(x^*, u^*) \in \mathcal{C}(I, I\!R^n) \times U$ with respect to b_0, b_1, η, ξ if there exist functions $b_0(x, x^*, u, u^*) \in I\!R_+$, $b_1(x, x^*, u, u^*) \in I\!R_+$, $\eta(t, x(t), x^*(t)) \in I\!R^n$ and $\xi(t, u(t), u^*(t)) \in I\!R^m$, with $\eta(t, x(t), x^*(t)) = 0$ if $x(t) = x^*(t)$, such that for all $(x, u) \in X \times U$,

$$\int_a^b \phi(t, x, \dot{x}, u)dt \leq \int_a^b \phi(t, x^*, \dot{x}^*, u^*)dt$$

$$\Longrightarrow b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t \phi_x(t, x^*, \dot{x}^*, u^*) + [D\eta]^t \phi_{\dot{x}}(t, x^*, \dot{x}^*, u^*)\right.$$

$$\left. + \xi^t \phi_u(t, x^*, \dot{x}^*, u^*)\right] dt \leq -\rho^1 \int_a^b d^2(t, x, x^*)dt,$$

$$- \int_a^b \psi(t, x^*, \dot{x}^*, u^*)dt \leqq 0$$

$$\Longrightarrow b_1(x,x^*,u,u^*)\int_a^b \big[\eta^t\psi_x(t,x^*,\dot{x}^*,u^*)+[D\eta]^t\psi_{\dot{x}}(t,x^*,\dot{x}^*,u^*)$$

$$+\xi^t\psi_u(t,x^*,u^*)\big]dt \leqq -\rho^2\int_a^b d^2(t,x,x^*)dt.$$

Definition 6 A pair (ϕ,ψ) is said to be weak strictly (B,ρ)-pseudo-type I at $(x^*,u^*)\in \mathcal{C}(I,I\!R^n)\times U$ with respect to b_0, b_1, η, ξ if there exist functions $b_0(x,x^*,u,u^*)\in I\!R_+$, $b_1(x,x^*,u,u^*)\in I\!R_+$, $\eta(t,x(t),x^*(t))\in I\!R^n$ and $\xi(t,u(t),u^*(t))\in I\!R^m$, with $\eta(t,x(t),x^*(t))=0$ if $x(t)=x^*(t)$, such that for all $(x,u)\in X\times U$,

$$\int_a^b \phi(t,x,\dot{x},u)dt \leq \int_a^b \phi(t,x^*,\dot{x}^*,u^*)dt$$

$$\Longrightarrow b_0(x,x^*,u,u^*)\int_a^b \big[\eta^t\phi_x(t,x^*,\dot{x}^*,u^*)+[D\eta]^t\,\phi_{\dot{x}}(t,x^*,\dot{x}^*,u^*)$$

$$+\xi^t\phi_u(t,x^*,\dot{x}^*,u^*)\big]dt < -\rho^1\int_a^b d^2(t,x,x^*)dt,$$

$$-\int_a^b \psi(t,x^*,\dot{x}^*,u^*)dt \leqq 0$$

$$\Longrightarrow b_1(x,x^*,u,u^*)\int_a^b \big[\eta^t\psi_x(t,x^*,\dot{x}^*,u^*)+[D\eta]^t\,\psi_{\dot{x}}(t,x^*,\dot{x}^*,u^*)$$

$$+\xi^t\psi_u(t,x^*,\dot{x}^*,u^*)\big]dt < -\rho^2\int_a^b d^2(t,x,x^*)dt.$$

3 Sufficient conditions

Chandra et al. [3] gave the Fritz-John necessary optimality conditions for the existence of an extremal solution for the single objective control problem. Using the relationship between the efficient solution of the problem (MCP) and the optimal solution of the associated scalar control problem, Gramatovici derived the necessary optimality conditions for the multiobjective control problems; details can be found in [6] and [8]. This method was used by Bhatia and Mehra in [2] for a multiobjective variational programming problem. Mond and Hanson [9] pointed out that if the solution for the problem (MCP) is normal, then the Fritz-John conditions reduce to Kuhn–Tucker conditions [8].

In this section, we establish various sufficient optimality conditions for (MCP) under (B,ρ)-type I and generalized (B,ρ)-type I conditions.

Theorem 1 *Let (x^*,u^*) be a feasible solution for (MCP) and assume there exist $\lambda_0^*\in I\!R^p$, $\lambda_0^*>0$ and piecewise smooth functions $\lambda^*: I\longrightarrow I\!R^m$, $\mu^*: I\longrightarrow I\!R^n$ such that for all $t\in I$,*

$$\lambda_0^{*t} f_x(t, x^*, u^*) + \lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*) + \dot{\mu^*}(t) = 0, t \in I, \quad (2)$$

$$\lambda_0^{*t} f_u(t, x^*, u^*) + \lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*) = 0, t \in I, \quad (3)$$

$$\lambda^*(t)^t g(t, x^*, u^*) = 0, \quad t \in I, \quad (4)$$

$$\lambda^*(t) \geqq 0, \quad t \in I. \quad (5)$$

Further, suppose that $(f, \lambda^*(t)^t g + \mu^*(t)^t (h - \dot{x}))$ *is* (B, ρ)*-type I at* (x^*, u^*) *with respect to* b_0, b_1, η, ξ *with* $b_0(x, x^*, u, u^*) > 0$ *for all* $(x, u) \in X \times U$ *and* $\lambda_0^{*t}\rho^1 + \rho^2 \geqq 0$, *then* (x^*, u^*) *is a proper efficient solution of* (MCP) *and therefore it is an efficient solution of* (MCP).

Proof Because $(f, \lambda^*(t)^t g + \mu^*(t)^t (h - \dot{x}))$ is (B, ρ)-type I at (x^*, u^*) with respect to b_0, b_1, η, ξ, therefore

$$b_0(x, x^*, u, u^*) \left[\int_a^b f(t, x, u)dt - \int_a^b f(t, x^*, u^*)dt \right]$$
$$\geqq \int_a^b \left[\eta^t f_x(t, x^*, u^*) + \xi^t f_u(t, x^*, u^*) \right] dt + \rho^1 \int_a^b d^2(t, x, x^*)dt, \quad (6)$$

$$-b_1(x, x^*, u, u^*) \int_a^b [\lambda^*(t)^t g(t, x^*, u^*) + \mu^*(t)^t (h(t, x^*, u^*) - \dot{x}^*)]dt$$
$$\geqq \int_a^b \left[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*) \right) + [D\eta]^t (-\mu^*(t)) \right.$$
$$\left. + \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*) \right) \right] dt + \rho^2 \int_a^b d^2(t, x, x^*)dt. \quad (7)$$

Multiplying (6) by the nonnegative vector λ_0^*, we get

$$b_0(x, x^*, u, u^*) \left[\int_a^b \lambda_0^{*t} f(t, x, u)dt - \int_a^b \lambda_0^{*t} f(t, x^*, u^*)dt \right]$$
$$\geqq \int_a^b \left[\eta^t \lambda_0^{*t} f_x(t, x^*, u^*) + \xi^t \lambda_0^{*t} f_u(t, x^*, u^*) \right] dt$$
$$+ \lambda_0^{*t} \rho^1 \int_a^b d^2(t, x, x^*)dt. \quad (8)$$

Since (x^*, u^*) is feasible for (MCP), then, $h(t, x^*, u^*) - \dot{x}^* = 0$ combined with (4), (7) can be rewritten as

$$0 \geqq \int_a^b \left[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) + [D\eta]^t(-\mu^*(t))\right.$$
$$\left. + \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\right] dt + \rho^2 \int_a^b d^2(t, x, x^*) dt,$$

it follows that

$$0 \geqq \int_a^b \eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) dt + \int_a^b \eta \dot{\mu}^*(t) dt$$
$$- [\eta^t \mu^*(t)]_{t=a}^{t=b} + \int_a^b \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right) dt$$
$$+ \rho^2 \int_a^b d^2(t, x, x^*) dt. \tag{9}$$

Adding (8) and (9), we obtain

$$b_0(x, x^*, u, u^*) \left[\int_a^b \lambda_0^{*t} f(t, x, u) dt - \int_a^b \lambda_0^{*t} f(t, x^*, u^*) dt\right]$$
$$\geqq \int_a^b \eta^t \left[\lambda_0^{*t} f_x(t, x^*, u^*) + \lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*) + \dot{\mu}^*(t)\right]$$
$$+ \xi^t \left[\lambda_0^{*t} f_u(t, x^*, u^*) + \lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right] dt$$
$$+ (\lambda_0^{*t} \rho^1 + \rho^2) \int_a^b d^2(t, x, x^*) dt. \tag{10}$$

Inequality (10) along with (2) and (3) yields

$$b_0(x, x^*, u, u^*) \left[\int_a^b \lambda_0^{*t} f(t, x, u) dt - \int_a^b \lambda_0^{*t} f(t, x^*, u^*) dt\right]$$
$$\geqq (\lambda_0^{*t} \rho^1 + \rho^2) \int_a^b d^2(t, x, x^*) dt.$$

Since $\lambda_0^{*t} \rho^1 + \rho^2 \geqq 0$, we get

$$b_0(x, x^*, u, u^*) \left[\int_a^b \lambda_0^{*t} f(t, x, u) dt - \int_a^b \lambda_0^{*t} f(t, x^*, u^*) dt\right] \geqq 0. \tag{11}$$

Because $b_0(x, x^*, u, u^*) > 0$ for all $(x, u) \in X \times U$, (11) gives

$$\int_a^b \lambda_0^{*t} f(t, x, u)dt \geqq \int_a^b \lambda_0^{*t} f(t, x^*, u^*)dt,$$

which implies that (x^*, u^*) minimizes $\int_a^b \lambda_0^{*t} f(t, x, u)dt$ over $X \times U$ with $\lambda_0^* > 0$. Hence, (x^*, u^*) is a properly efficient solution for (MCP) on account of Theorem 2 (see [2]) and therefore (x^*, u^*) is an efficient solution of (MCP). □

Theorem 2 *Let (x^*, u^*) be a feasible solution for (MCP) and assume there exist $\lambda_0^* \in I\!R^p$, $\lambda_0^* \geq 0$ and piecewise smooth functions $\lambda^* : I \longrightarrow I\!R^m$, $\mu^* : I \longrightarrow I\!R^n$ such that for all $t \in I$, $(x^*, u^*, \lambda_0^*, \lambda^*)$ satisfy (2)–(5) of Theorem 1. Further, suppose that $(f, \lambda^*(t)^t g + \mu^*(t)^t(h - \dot{x}))$ is semistrictly (B, ρ)-type I at (x^*, u^*) with respect to b_0, b_1, η, ξ with $\lambda_0^{*t}\rho^1 + \rho^2 \geqq 0$, then (x^*, u^*) is an efficient solution of (MCP).*

Proof If (x^*, u^*) is not an efficient solution of (MCP), then there exists an $(x, u) \in X \times U$ such that

$$\int_a^b f(t, x, u)dt \leq \int_a^b f(t, x^*, u^*)dt.$$

Since $b_0(x, x^*, u, u^*) \geqq 0$, we obtain

$$b_0(x, x^*, u, u^*)\left[\int_a^b f(t, x, u)dt - \int_a^b f(t, x^*, u^*)dt\right] \leqq 0. \tag{12}$$

Since (x^*, u^*) is feasible for (MCP), then, $h(t, x^*, u^*) - \dot{x}^* = 0$ combined with (4), we have

$$-b_1(x, x^*, u, u^*)\int_a^b \lambda^*(t)^t g(t, x^*, u^*) + \mu^*(t)^t(h(t, x^*, u^*) - \dot{x}^*))dt \leqq 0. \tag{13}$$

Equations 12 and 13 with the fact that $(f, \lambda^*(t)^t g + \mu^*(t)^t(h - \dot{x}))$ is semistrictly (B, ρ)-type I at (x^*, u^*) with respect to b_0, b_1, η, ξ, give us

$$\int_a^b \left[\eta^t f_x(t, x^*, u^*) + \xi^t f_u(t, x^*, u^*)\right]dt < -\rho^1 \int_a^b d^2(t, x, x^*)dt, \tag{14}$$

$$\int_a^b \Big[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) + [D\eta]^t(-\mu^*(t))$$
$$+\xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\Big]dt \leqq -\rho^2 \int_a^b d^2(t, x, x^*)dt. \tag{15}$$

Multiplying (14) by $\lambda_0^* \geq 0$, we get

$$\int_a^b \left[\eta^t \lambda_0^{*t} f_x(t, x^*, u^*) + \xi^t \lambda_0^{*t} f_u(t, x^*, u^*)\right] dt < -\lambda_0^{*t} \rho^1 \int_a^b d^2(t, x, x^*) dt. \tag{16}$$

Equation 15 gives

$$0 \geqq \int_a^b \eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) dt + \int_a^b \eta \dot{\mu}^*(t) dt - [\eta^t \mu^*(t)]_{t=a}^{t=b} + \int_a^b \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right) dt + \rho^2 \int_a^b d^2(t, x, x^*) dt. \tag{17}$$

Adding (16) and (17), we get

$$\int_a^b \eta^t \left[\lambda_0^{*t} f_x(t, x^*, u^*) + \lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*) + \dot{\mu}^*(t)\right] + \xi^t \left[\lambda_0^{*t} f_u(t, x^*, u^*) + \lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right] dt < -(\lambda_0^{*t} \rho^1 + \rho^2) \int_a^b d^2(t, x, x^*) dt \leqq 0,$$

which contradicts (2) and (3). Hence (x^*, u^*) is an efficient solution for (MCP) and the proof is complete. □

Theorem 3 *Let (x^*, u^*) be a feasible solution for (MCP) and assume there exist $\lambda_0^* \in I\!R^p$, $\lambda_0^* > 0$ and piecewise smooth functions $\lambda^* : I \longrightarrow I\!R^m$, $\mu^* : I \longrightarrow I\!R^n$ such that for all $t \in I$, $(x^*, u^*, \lambda_0^*, \lambda^*)$ satisfy (2)–(5) of Theorem 1. Further, suppose that $(f, \lambda^*(t)^t g + \mu^*(t)^t (h - \dot{x}))$ is strong (B, ρ)-pseudo-quasi-type I at (x^*, u^*) with respect to b_0, b_1, η, ξ with $b_1(x, x^*, u, u^*) > 0$ and $\lambda_0^{*t} \rho^1 + \frac{b_0(x, x^*, u, u^*) \rho^2}{b_1(x, x^*, u, u^*)} \geqq 0$ for all $(x, u) \in X \times U$, then (x^*, u^*) is an efficient solution of (MCP).*

Proof If (x^*, u^*) is not an efficient solution of (MCP), then there exists an $(x, u) \in X \times U$ such that

$$\int_a^b f(t, x, u) dt \leq \int_a^b f(t, x^*, u^*) dt.$$

Since (x^*, u^*) is feasible for (MCP), then, $h(t, x^*, u^*) - \dot{x}^* = 0$ combined with (4), we have

$$-\int_a^b \lambda^*(t)^t g(t, x^*, u^*) + \mu^*(t)^t (h(t, x^*, u^*) - \dot{x}^*))dt \leqq 0.$$

Since $(f, \lambda^*(t)^t g + \mu^*(t)^t (h - \dot{x}))$ is strong (B, ρ)-pseudo-quasi B-type I at (x^*, u^*) with respect to b_0, b_1, η, ξ, therefore

$$b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t f_x(t, x^*, u^*) + \xi^t f_u(t, x^*, u^*)\right] dt$$

$$\leq -\rho^1 \int_a^b d^2(t, x, x^*)dt,$$

$$b_1(x, x^*, u, u^*) \int_a^b \left[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) + [D\eta]^t(-\mu^*(t))\right.$$

$$\left. + \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\right] dt$$

$$\leqq -\rho^2 \int_a^b d^2(t, x, x^*)dt.$$

Since $\lambda_0^* > 0$ and $b_1(x, x^*, u, u^*)$ is positive, we get

$$b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t \lambda_0^{*t} f_x(t, x^*, u^*) + \xi^t \lambda_0^t f_u(t, x^*, u^*)\right] dt$$

$$< -\lambda_0^{*t} \rho^1 \int_a^b d^2(t, x, x^*)dt, \tag{18}$$

$$\int_a^b \left[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) + [D\eta]^t(-\mu^*(t))\right.$$

$$\left. + \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\right] dt$$

$$\leqq -\frac{\rho^2}{b_1(x, x^*, u, u^*)} \int_a^b d^2(t, x, x^*)dt. \tag{19}$$

Multiplying (19) by $b_0(x, x^*, u, u^*) \geqq 0$, we get

$$b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right)\right.$$

$$\left. + [D\eta]^t(-\mu^*(t)) + \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\right] dt$$

$$\leqq -\frac{b_0(x, x^*, u, u^*)\rho^2}{b_1(x, x^*, u, u^*)} \int_a^b d^2(t, x, x^*)dt. \tag{20}$$

Equation 20 gives

$$b_0(x,x^*,u,u^*)\int_a^b \eta^t\left(\lambda^*(t)^t g_x(t,x^*,u^*)+\mu^*(t)^t h_x(t,x^*,u^*)\right)dt$$
$$+\int_a^b \eta\dot{\mu}^*(t)dt-[\eta^t\mu^*(t)]_{t=a}^{t=b}+\int_a^b \xi^t\left(\lambda^*(t)^t g_u(t,x^*,u^*)+\mu^*(t)^t h_u(t,x^*,u^*)\right)dt$$
$$\leqq -\frac{b_0(x,x^*,u,u^*)\rho^2}{b_1(x,x^*,u,u^*)}\int_a^b d^2(t,x,x^*)dt. \tag{21}$$

Adding (18) and (21), we obtain

$$b_0(x,x^*,u,u^*)$$
$$\int_a^b \eta^t\times\left[\lambda_0^{*t}f_x(t,x^*,u^*)+\lambda^*(t)^t g_x(t,x^*,u^*)+\mu^*(t)^t h_x(t,x^*,u^*)+\dot{\mu}^*(t)\right]$$
$$+\xi^t\left[\lambda_0^{*t}f_u(t,x^*,u^*)+\lambda^*(t)^t g_u(t,x^*,u^*)+\mu^*(t)^t h_u(t,x^*,u^*)\right]dt$$
$$<-\left(\lambda_0^{*t}\rho^1+\frac{b_0(x,x^*,u,u^*)\rho^2}{b_1(x,x^*,u,u^*)}\right)\int_a^b d^2(t,x,x^*)dt\leqq 0,$$

which contradicts (2) and (3). Hence (x^*,u^*) is an efficient solution for (MCP) and the proof is complete. □

In the next theorem, we replace the strong (B,ρ)-pseudo-quasi-type I by the weak strictly (B,ρ)-pseudo-quasi B-type I of $(f,\lambda^*(t)^t g+\mu^*(t)^t(h-\dot{x}))$.

Theorem 4 *Let (x^*,u^*) be a feasible solution for (MCP) and assume there exist $\lambda_0^*\in I\!R^p$, $\lambda_0^*\geq 0$ and piecewise smooth functions $\lambda^*:I\longrightarrow I\!R^m$, $\mu^*:I\longrightarrow I\!R^n$ such that for all $t\in I$, $(x^*,u^*,\lambda_0^*,\lambda^*)$ satisfy (2)–(5) of Theorem 1. Further, suppose that $(f,\lambda^*(t)^t g+\mu^*(t)^t(h-\dot{x}))$ is weak strictly (B,ρ)-pseudo-quasi B-type I at (x^*,u^*) with respect to b_0, b_1, η, ξ with $b_1(x,x^*,u,u^*)>0$ and $\lambda_0^{*t}\rho^1+\frac{b_0(x,x^*,u,u^*)\rho^2}{b_1(x,x^*,u,u^*)}\geqq 0$ for all $(x,u)\in X\times U$, then (x^*,u^*) is an efficient solution of (MCP).*

Proof If (x^*,u^*) is not an efficient solution of (MCP), then there exists an $(x,u)\in X\times U$ such that

$$\int_a^b f(t,x,u)dt\leq\int_a^b f(t,x^*,u^*)dt.$$

Since (x^*,u^*) is feasible for (MCP), then, $h(t,x^*,u^*)-\dot{x}^*=0$ combined with (4), we have

$$-\int_a^b \lambda^*(t)^t g(t,x^*,u^*)+\mu^*(t)^t(h(t,x^*,u^*)-\dot{x}^*))dt\leqq 0.$$

Since $(f, \lambda^*(t)^t g + \mu^*(t)^t(h - \dot{x}))$ is weak strictly (B, ρ) pseudo-quasi-type I at (x^*, u^*) with respect to b_0, b_1, η, ξ, therefore

$$b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t f_x(t, x^*, u^*) + \xi^t f_u(t, x^*, u^*)\right] dt$$

$$< -\rho^1 \int_a^b d^2(t, x, x^*) dt, \tag{22}$$

$$b_1(x, x^*, u, u^*) \int_a^b \Big[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) + [D\eta]^t(-\mu^*(t))$$
$$+ \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\Big] dt$$
$$\leqq -\rho^2 \int_a^b d^2(t, x, x^*) dt.$$

Multiplying (22) by $\lambda_0^* \geq 0$, we get

$$b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t \lambda_0^{*t} f_x(t, x^*, u^*) + \xi^t \lambda_0^t f_u(t, x^*, u^*)\right] dt$$

$$< -\lambda_0^{*t} \rho^1 \int_a^b d^2(t, x, x^*) dt,$$

and now the proof is similar to that of Theorem 3. □

In our final sufficiency result below, we invoke the weak strictly (B, ρ)-pseudo-type I of $(f, \lambda^*(t)^t g + \mu^*(t)^t(h - \dot{x}))$.

Theorem 5 *Let (x^*, u^*) be a feasible solution for (MCP) and assume there exist $\lambda_0^* \in I\!R^p$, $\lambda_0^* \geqq 0$ and piecewise smooth functions $\lambda^* : I \longrightarrow I\!R^m$, $\mu^* : I \longrightarrow I\!R^n$ such that for all $t \in I$, $(x^*, u^*, \lambda_0^*, \lambda^*)$ satisfy (2)–(5) of Theorem 1. Further, suppose that $(f, \lambda^*(t)^t g + \mu^*(t)^t(h - \dot{x}))$ is weak strictly (B, ρ)-pseudo B-type I at (x^*, u^*) with respect to b_0, b_1, η, ξ with $\frac{\lambda_0^{*t}\rho^1}{b_0(x,x^*,u,u^*)} + \frac{\rho^2}{b_1(x,x^*,u,u^*)} \geqq 0$ for all $(x, u) \in X \times U$, then (x^*, u^*) is an efficient solution of (MCP).*

Proof If (x^*, u^*) is not an efficient solution of (MCP), then there exists an $(x, u) \in X \times U$ such that

$$\int_a^b f(t, x, u) dt \leq \int_a^b f(t, x^*, u^*) dt.$$

Since (x^*, u^*) is feasible for (MCP), then, $h(t, x^*, u^*) - \dot{x}^* = 0$ combined with (4), we have

$$-\int_a^b \lambda^*(t)^t g(t, x^*, u^*) + \mu^*(t)^t (h(t, x^*, u^*) - \dot{x}^*)) dt \leqq 0.$$

Since $(f, y^*(t)^t g)$ is weak strictly (B, ρ)-pseudo-type I at x^* with respect to b_0, b_1, η, ξ, therefore

$$b_0(x, x^*, u, u^*) \int_a^b \left[\eta^t f_x(t, x^*, u^*) + \xi^t \lambda_0^t f_u(t, x^*, u^*)\right] dt < -\rho^1 \int_a^b d^2(t, x, x^*) dt, \tag{23}$$

$$b_1(x, x^*, u, u^*) \int_a^b \left[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) + [D\eta]^t (-\mu^*(t)) + \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\right] dt < -\rho^2 \int_a^b d^2(t, x, x^*) dt. \tag{24}$$

From (23) and (24), we have $b_0(x, x^*, u, u^*) \neq 0$ and $b_1(x, x^*, u, u^*) \neq 0$, which imply that

$$\int_a^b \left[\eta^t f_x(t, x^*, u^*) + \xi^t f_u(t, x^*, u^*)\right] dt < -\frac{\rho^1}{b_0(x, x^*, u, u^*)} \int_a^b d^2(t, x, x^*) dt, \tag{25}$$

$$\int_a^b \left[\eta^t \left(\lambda^*(t)^t g_x(t, x^*, u^*) + \mu^*(t)^t h_x(t, x^*, u^*)\right) + [D\eta]^t (-\mu^*(t)) + \xi^t \left(\lambda^*(t)^t g_u(t, x^*, u^*) + \mu^*(t)^t h_u(t, x^*, u^*)\right)\right] dt < -\frac{\rho^2}{b_1(x, x^*, u, u^*)} \int_a^b d^2(t, x, x^*) dt. \tag{26}$$

Multiplying (25) by $\lambda_0^* \geqq 0$, we get

$$\int_a^b \left[\eta^t \lambda_0^{*t} f_x(t, x^*, u^*) + \xi^t \lambda_0^t f_u(t, x^*, u^*)\right] dt \leqq -\frac{\lambda_0^{*t} \rho^1}{b_0(x, x^*, u, u^*)} \int_a^b d^2(t, x, x^*) dt. \tag{27}$$

Equation 26 gives

$$\int_a^b \eta^t \left(\lambda^*(t)^t g_x(t,x^*,u^*) + \mu^*(t)^t h_x(t,x^*,u^*)\right) dt + \int_a^b \eta\dot{\mu}^*(t)dt$$
$$- [\eta^t \mu^*(t)]_{t=a}^{t=b} + \int_a^b \xi^t \left(\lambda^*(t)^t g_u(t,x^*,u^*) + \mu^*(t)^t h_u(t,x^*,u^*)\right) dt$$
$$\leqq -\frac{\rho^2}{b_1(x,x^*,u,u^*)} \int_a^b d^2(t,x,x^*)dt. \tag{28}$$

Adding (27) and (28), we get

$$\int_a^b \eta^t \left[\lambda_0^{*t} f_x(t,x^*,u^*) + \lambda^*(t)^t g_x(t,x^*,u^*) + \mu^*(t)^t h_x(t,x^*,u^*) + \dot{\mu}^*(t)\right]$$
$$+ \xi^t \left[\lambda_0^{*t} f_u(t,x^*,u^*) + \lambda^*(t)^t g_u(t,x^*,u^*) + \mu^*(t)^t h_u(t,x^*,u^*)\right] dt$$
$$< -\left(\frac{\lambda^{*t}\rho^1}{b_0(x,x^*,u,u^*)} + \frac{\rho^2}{b_1(x,x^*,u,u^*)}\right) \int_a^b d^2(t,x,x^*)dt \leqq 0,$$

which contradicts (2) and (3). Hence the result. □

4 Mixed type duality

Let J_1 be a subset of M, $J_2 = M \setminus J_1$, and e be the vector of $I\!R^p$ whose components are all ones. We consider the following mixed type dual for (MCP),

$$(XMCP) \text{ Maximize} \int_a^b \{f(t,y,v) + [\lambda_{J_1}(t)^t g_{J_1}(t,y,v)]e\}dt$$

subject to

$$y(a) = \alpha, \quad y(b) = \beta,$$
$$\lambda_0^t f_y(t,y,v) + \lambda(t)^t g_y(t,y,v) + \mu(t)^t h_y(t,y,v) + \dot{\mu}(t) = 0, \quad t \in I, \tag{29}$$
$$\lambda_0^t f_v(t,y,v) + \lambda(t)^t g_v(t,y,v) + \mu(t)^t h_v(t,y,v) = 0, \quad t \in I, \tag{30}$$
$$\int_a^b \mu(t)^t [h(t,y,v) - \dot{y}]dt \geqq 0, \tag{31}$$
$$\int_a^b \lambda_{J_2}(t)^t g_{J_2}(t,y,v)dt \geqq 0, \tag{32}$$
$$\lambda(t) \geqq 0, \quad t \in I \tag{33}$$
$$\lambda_0 \geqq 0, \quad \lambda_0^t e = 1. \tag{34}$$

We note that we get a Mond–Weir dual [8] for $J_1 = \emptyset$ and a Wolfe dual [1] for $J_2 = \emptyset$ in $(XMCP)$, respectively.
We shall prove various duality results for (MCP) and $(XMCP)$ under generalized (B, ρ)-type I conditions.

Theorem 6 *(*Weak duality*) If for all feasible (x, u) of (MCP) and all feasible $(y, v, \lambda_0, \lambda, \mu)$ of $(XMCP)$, any of the following conditions holds:*

(a) $\lambda_0 > 0$, $(f + \lambda_{J_1}(t)^t g_{J_1} e, \lambda_{J_2}(t)^t g_{J_2} + \mu(t)^t (h - \dot{y}))$ *is strong (B, ρ)-pseudo-quasi-type I at (y, v) with respect to b_0, b_1, η, ξ with $b_1(x, y, u, v) > 0$ and $\lambda_0^t \rho^1 + \frac{b_0(x,y,u,v)\rho^2}{b_1(x,y,u,v)} \geqq 0$ for all $(x, u) \in X \times U$,*

(b) $(f + \lambda_{J_1}(t)^t g_{J_1} e, \lambda_{J_2}(t)^t g_{J_2} + \mu(t)^t (h - \dot{y}))$ *is weak strictly (B, ρ)-pseudo-quasi-type I at (y, v) with respect to b_0, b_1, η, ξ with $b_1(x, y, u, v) > 0$ and $\lambda_0^t \rho^1 + \frac{b_0(x,y,u,v)\rho^2}{b_1(x,y,u,v)} \geqq 0$ for all $(x, u) \in X \times U$,*

(c) $(f + \lambda_{J_1}(t)^t g_{J_1} e, \lambda_{J_2}(t)^t g_{J_2} + \mu(t)^t (h - \dot{y}))$ *is weak strictly (B, ρ)-pseudo-type I at (y, v) with respect to b_0, b_1, η, ξ and $\frac{\lambda_0^t \rho^1}{b_0(x,y,u,v)} + \frac{\rho^2}{b_1(x,y,u,v)} \geqq 0$, for all $(x, u) \in X \times U$, then the following cannot hold*

$$\int_a^b f(t, x, u)dt \leq \int_a^b \{f(t, y, v) + [\lambda_{J_1}(t)^t g_{J_1}(t, y, v)]e\}dt.$$

Proof Let (x, u) be feasible for (MCP) and $(y, v, \lambda_0, \lambda, \mu)$ be feasible for $(XMCP)$. Suppose that

$$\int_a^b f(t, x, u)dt \leq \int_a^b \{f(t, y, v) + [\lambda_{J_1}(t)^t g_{J_1}(t, y, v)]e\}dt.$$

Since (x, u) is feasible for (MCP) and $(y, v, \lambda_0, \lambda, \mu)$ is feasible for $(XMCP)$ we have

$$\int_a^b \{f(t, x, u) + [\lambda_{J_1}(t)^t g_{J_1}(t, x, u)]e\}dt$$
$$\leq \int_a^b \{f(t, y, v) + [\lambda_{J_1}(t)^t g_{J_1}(t, y, v)]e\}dt. \tag{35}$$

From (32) and (31), we have

$$-\int_a^b [\lambda_{J_2}(t)^t g_{J_2}(t, y, v) + \mu(t)^t (h(t, y, v) - \dot{y})]dt \leqq 0. \tag{36}$$

If (a) holds, then

$$
\begin{aligned}
&b_0(x, y, u, v) \int_a^b \eta^t \left[f_y(t, y, v) + e\lambda_{J_1}(t)^t g_{J_{1y}}(t, y, v)\right] \\
&\quad + \xi^t \left[f_v(t, y, v) + e\lambda_{J_1}(t)^t g_{J_{1v}}(t, y, v)\right] dt \\
&\quad \leq -\rho^1 \int_a^b d^2(t, x, y) dt,
\end{aligned}
$$

$$
\begin{aligned}
&b_1(x, y, u, v) \int_a^b \left[\eta^t \left(\lambda_{J_2}(t)^t g_{J_2 y}(t, y, v) + \mu(t)^t h_y(t, y, v)\right)\right. \\
&\quad \left. + [D\eta]^t (-\mu(t)) + \xi^t \left(\lambda_{J_2}(t)^t g_{J_2 v}(t, y, v) + \mu(t)^t h_v(t, y, v)\right)\right] dt \\
&\quad \leqq -\rho^2 \int_a^b d^2(t, x, y) dt.
\end{aligned}
$$

Since $b_1(x, y, u, v)$ is positive and $\lambda_0 > 0$, we get

$$
\begin{aligned}
&b_0(x, y, u, v) \int_a^b \eta^t \left[\lambda_0^t f_y(t, y, v) + \lambda_{J_1}(t)^t g_{J_{1y}}(t, y, v)\right. \\
&\quad \left. + \xi^t \left(\lambda_0^t f_v(t, y, v) + \lambda_{J_1}(t)^t g_{J_{1v}}(t, y, v)\right)\right] dt \\
&\quad < -\lambda_0^t \rho^1 \int_a^b d^2(t, x, y) dt,
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&\int_a^b \left[\eta^t \left(\lambda_{J_2}(t)^t g_{J_2 y}(t, y, v) + \mu(t)^t h_y(t, y, v)\right) + [D\eta]^t (-\mu(t))\right. \\
&\quad \left. + \xi^t \left(\lambda_{J_2}(t)^t g_{J_2 v}(t, y, v) + \mu(t)^t h_v(t, y, v)\right)\right] dt \\
&\quad \leqq -\frac{\rho^2}{b_1(x, y, u, v)} \int_a^b d^2(t, x, y) dt.
\end{aligned}
\tag{38}
$$

Multiplying (38) by $b_0(x, y, u, v) \geqq 0$, we get

$$
\begin{aligned}
&b_0(x, y, u, v) \int_a^b \left[\eta^t \left(\lambda_{J_2}(t)^t g_{J_2 y}(t, y, v) + \mu(t)^t h_y(t, y, v)\right)\right. \\
&\quad \left. + [D\eta]^t (-\mu(t)) + \xi^t \left(\lambda_{J_2}(t)^t g_{J_2 v}(t, y, v) + \mu(t)^t h_v(t, y, v)\right)\right] dt \\
&\quad \leqq -\frac{b_0(x, y, u, v)\rho^2}{b_1(x, y, u, v)} \int_a^b d^2(t, x, y) dt.
\end{aligned}
\tag{39}
$$

Equation 39 gives

$$b_0(x, y, u, v) \int_a^b \eta^t \left(\lambda_{J_2}(t)^t g_{J_2 y}(t, y, v) + \mu(t)^t h_y(t, y, v)\right) dt + \int_a^b \eta \dot{\mu}(t) dt$$
$$- [\eta^t \mu(t)]_{t=a}^{t=b} + \int_a^b \xi^t \left(\lambda_{J_2}(t)^t g_{J_2 v}(t, y, v) + \mu(t)^t h_v(t, y, v)\right) dt$$
$$\leqq -\frac{b_0(x, y, u, v)\rho^2}{b_1(x, y, u, v)} \int_a^b d^2(t, x, y) dt. \tag{40}$$

Adding (37) and (40), we obtain

$$b_0(x, y, u, v) \int_a^b \eta^t \left[\lambda_0^t f_y(t, y, v) + \lambda(t)^t g_y(t, y, v) + \mu(t)^t h_y(t, y, v) + \dot{\mu}(t)\right]$$
$$+ \xi^t \left[\lambda_0^t f_v(t, y, v) + \lambda(t)^t g_v(t, y, v) + \mu(t)^t h_v(t, y, v)\right] dt$$
$$< -\left(\lambda_0^t \rho^1 + \frac{b_0(x, y, u, v)\rho^2}{b_1(x, y, u, v)}\right) \int_a^b d^2(t, x, y) dt \leqq 0,$$

which contradicts (29) and (30).
Now, by hypothesis (b) and from (32) and (31), we get

$$b_0(x, y, u, v) \int_a^b \eta^t \left[f_y(t, y, v) + e\lambda_{J_1}(t)^t g_{J_{1y}}(t, y, v)\right.$$
$$\left. + \xi^t \left[f_v(t, y, v) + e y_{J_1}(t)^t g_{J_{1v}}(t, y, v)\right)\right] dt$$
$$< -\rho^1 \int_a^b d^2(t, x, y) dt,$$

$$b_1(x, y, u, v) \int_a^b \left[\eta^t \left(\lambda_{J_2}(t)^t g_{J_2 y}(t, y, v) + \mu(t)^t h_y(t, y, v)\right) + [D\eta]^t(-\mu(t))\right.$$
$$\left. + \xi^t \left(\mu(t)^t g_{J_2 v}(t, y, v) + \mu(t)^t h_v(t, y, v)\right)\right] dt$$
$$\leqq -\rho^2 \int_a^b d^2(t, x, y) dt.$$

Since $b_1(x, y, u, v)$ is positive, $\lambda_0 \geq 0$ and $b_0(x, y, u, v) \geqq 0$, we get (38) again contradicting (29) and (30).

If (c) holds, then from (32) and (31), we get

$$b_0(x, y, u, v) \int_a^b \eta^t \left[f_y(t, y, v) + e\lambda_{J_1}(t)^t g_{J_{1y}}(t, y, v) \right]$$
$$+ \xi^t \left[f_v(t, y, v) + e\lambda_{J_1}(t)^t g_{J_{1v}}(t, y, v) \right] dt$$
$$< -\rho^1 \int_a^b d^2(t, x, y) dt, \tag{41}$$

$$b_1(x, y, u, v) \int_a^b \left[\eta^t \left(\lambda_{J_2}(t)^t g_{J_2 y}(t, y, v) + \mu(t)^t h_y(t, y, v) \right) \right.$$
$$\left. + [D\eta]^t (-\mu(t)) + \xi^t \left(\lambda_{J_2}(t)^t g_{J_2 v}(t, y, v) + \mu(t)^t h_v(t, y, v) \right) \right] dt$$
$$< -\rho^2 \int_a^b d^2(t, x, y) dt. \tag{42}$$

From (41) and (42), we have $b_0(x, y, u, v) \neq 0$ and $b_1(x, y, u, v) \neq 0$, which give

$$\int_a^b \eta^t \left[f_y(t, y, v) + e\lambda_{J_1}(t)^t g_{J_{1y}}(t, y, v) \right]$$
$$+ \xi^t \left[f_v(t, y, v) + e\lambda_{J_1}(t)^t g_{J_{1v}}(t, y, v) \right] dt$$
$$< -\frac{\rho^1}{b_0(x, y, u, v)} \int_a^b d^2(t, x, y) dt, \tag{43}$$

$$\int_a^b \left[\eta^t \left(\lambda_{J_2}(t)^t g_{J_2 y}(t, y, v) + \mu(t)^t h_y(t, y, v) \right) + [D\eta]^t (-\mu(t)) \right.$$
$$\left. + \xi^t \left(\lambda_{J_2}(t)^t g_{J_2 v}(t, y, v) + \mu(t)^t h_v(t, y, v) \right) \right] dt$$
$$< -\frac{\rho^2}{b_1(x, y, u, v)} \int_a^b d^2(t, x, y) dt. \tag{44}$$

Because $\lambda_0 \geq 0$, (43) gives

$$\int_a^b \eta^t [\lambda_0^t f_y(t, y, v) + \mu_{J_1}(t)^t g_{J_{1y}}(t, y, v)$$
$$+ \xi^t [\lambda_0^t f_v(t, y, v) + \lambda_{J_1}(t)^t g_{J_{1v}}(t, y, v))] dt$$
$$\leqq -\frac{\lambda_0^t \rho^1}{b_0(x, y, u, v)} \int_a^b d^2(t, x, y) dt. \tag{45}$$

Adding (44) and (45), we obtain

$$\int_a^b \eta^t \left[\lambda_0^t f_y(t,y,v) + \lambda(t)^t g_y(t,y,v) + \mu(t)^t h_y(t,y,v) + \dot{\mu}(t)\right]$$
$$+ \xi^t \left[\lambda_0^t f_v(t,y,v) + \lambda(t)^t g_v(t,y,v) + \mu(t)^t h_v(t,y,v)\right] dt$$
$$< -\left(\frac{\lambda_0^t \rho^1}{b_0(x,y,u,v)} + \frac{\rho^2}{b_1(x,y,u,v)}\right) \int_a^b d^2(t,x,y)dt \leqq 0,$$

which contradicts again (29) and (30). □

Corollary 1 *Let $(y^*, v^*, \lambda_0^*, \lambda^*, \mu^*)$ be a feasible solution for $(XMCP)$. Assume that $\lambda_{J_1}^*(t)^t g_{J_1}(t, y^*, v^*) = 0$ and assume that (y^*, v^*) is feasible for (MCP). If weak duality Theorem 6 holds between (MCP) and $(XMCP)$, then (y^*, v^*) is an efficient solution for (MCP) and $(y^*, v^*, \lambda_0^*, \lambda^*, \mu^*)$ is an efficient solution for $(XMCP)$.*

Theorem 7 (Strong Duality) *Let (x^*, u^*) be an efficient solution for (MCP) at which the Kuhn–Tucker qualification constraint is satisfied [6], then there exists $\lambda_0^* \in I\!R^p$, $\lambda_0^* \geq 0$, $\lambda_0^{*t} e = 1$ and piecewise smooth functions $\lambda^* : I \longrightarrow I\!R^m$ and $\mu^* : I \longrightarrow I\!R^n$ such that $(x^*, u^*, \lambda_0^*, \lambda^*, \mu^*)$ is feasible for $(XMCP)$ with $\lambda_{J_1}^*(t)^t g_{J_1}(t, x^*, u^*) = 0$.If also weak duality Theorem 6 holds between (MCP) and $(XMCP)$, then $(y^*, v^*, \lambda_0^*, \lambda^*, \mu^*)$ is an efficient solution for $(XMCP)$.*

Theorem 8 (Strict Converse Duality) *Let (x^*, u^*) be a feasible solution for (MCP) and let $(y^*, v^*, \lambda_0^*, \lambda^*, \mu^*)$ be a feasible solution for $(XMCP)$ such that*

$$\int_a^b \lambda_0^{*t} f(t, x^*, u^*)dt = \int_a^b \{\lambda_0^{*t} f(t, y^*, v^*) + \lambda_{J_1}^*(t)^t g_{J_1}(t, y^*, v^*)\}dt. \tag{46}$$

Further, let $\left(f + \lambda_{J_1}^*(t)^t g_{J_1} e, \lambda_{J_2}^*(t)^t g_{J_2} + \mu(t)^{*t}(h - \dot{y}^*)\right)$ be semistrictly (B, ρ)-type I at (y^*, v^*) with respect to b_0, b_1, η, ξ with $\lambda_0^{*t}\rho^1 + \lambda_0^{*t}\rho^2 \geqq 0$ then (x^*, u^*)=(y^*, v^*).

Proof Suppose, on the contrary, that $(x^*, u^*) \neq (y^*, v^*)$. Since $\left(f + \lambda_{J_1}^*(t)^t g_{J_1} e, \lambda_{J_2}^*(t)^t g_{J_2} + \mu(t)^{*t}(h - \dot{y}^*)\right)$ is semistrictly (B, ρ)-type I at (y^*, v^*), then

$$b_0(x^*, y^*, u^*, v^*) \left[\int_a^b \left\{f(t, x^*, u^*) + \left[\lambda_{J_1}^*(t)^t g_{J_1}(t, x^*, u^*)\right] e\right\} dt\right.$$
$$\left. - \int_a^b \{f(t, y^*, v^*) + [\lambda_{J_1}^*(t)^t g_{J_1}(t, y^*, v^*)]e\}dt\right]$$
$$> \int_a^b \eta^t \left[f_y(t, y^*, v) + e\lambda_{J_1}^*(t)^t g_{J_{1y}}(t, y^*, v^*)\right]$$
$$+ \xi^t \left[f_v(t, y^*, v^*) + e\lambda_{J_1}(t)^t g_{J_{1v}}(t, y^*, v^*)\right] dt$$

$$+\rho^1 \int_a^b d^2(t, x^*, y^*)dt. \tag{47}$$

Since $\lambda_0^* \geq 0$, it follows that

$$\begin{aligned}
&b_0(x^*, y^*, u^*, v^*)\left[\int_a^b \{\lambda_0^{*t} f(t, x^*, u^*) + \lambda_{J_1}^*(t)^t g_{J_1}(t, x^*, u^*)\}dt\right.\\
&\left.-\int_a^b \{\lambda_0^{*t} f(t, y^*, v^*) + \lambda_{J_1}^*(t)^t g_{J_1}(t, y^*, v^*)\}dt\right]\\
&> \int_a^b \eta^t \left[\lambda_0^{*t} f_y(t, y^*, v^*) + \lambda_{J_1}^*(t)^t g_{J_{1y}}(t, y^*, v^*)\right.\\
&\left.+\xi^t \left[\lambda_0^{*t} f_v(t, y^*, v^*) + y_{J_1}(t)^t g_{J_{1v}}(t, y^*, v^*)\right)\right] dt\\
&+\lambda_0^{*t} \rho^1 \int_a^b d^2(t, x^*, y^*)dt.
\end{aligned}$$

Equations 46, 29 and 30 give

$$\begin{aligned}
&b_0(x^*, y^*, u^*, v^*)\left[\int_a^b \lambda_{J_1}^*(t)^t g_{J_1}(t, x^*, u^*)dt\right]\\
&> -\int_a^b \left[\eta^t \left(\lambda_{J_2}^*(t)^t g_{J_2 y}(t, y^*, v^*) + \mu^*(t)^t h_y(t, y^*, v^*) + \dot{\mu}^*(t)\right)\right.\\
&\left.+\xi^t \left(\lambda_{J_2}^*(t)^t g_{J_2 v}(t, y^*, v^*) + \mu^*(t)^t h_v(t, y^*, v^*)\right)\right] dt\\
&+\lambda_0^{*t} \rho^1 \int_a^b d^2(t, x^*, y^*)dt.
\end{aligned}$$

Since $\lambda_0^{*t} \rho^1 + \lambda_0^{*t} \rho^2 \geqq 0$, we get

$$\begin{aligned}
&b_0(x^*, y^*, u^*, v^*)\left[\int_a^b \lambda_{J_1}^*(t)^t g_{J_1}(t, x^*, u^*)dt\right]\\
&> -\int_a^b \left[\eta^t \left(\lambda_{J_2}^*(t)^t g_{J_2 y}(t, y^*, v^*) + \mu^*(t)^t h_y(t, y^*, v^*) + \dot{\mu}^*(t)\right)\right.\\
&\left.+\xi^t \left(\lambda_{J_2}^*(t)^t g_{J_2 v}(t, y^*, v^*) + \mu^*(t)^t h_v(t, y^*, v^*)\right)\right] dt - \lambda_0^{*t} \rho^2 \int_a^b d^2(t, x^*, y^*)dt.
\end{aligned}$$

Using the hypothesis on $(f + \lambda^*_{J_1}(t)^t g_{J_1} e, \lambda^*_{J_2}(t)^t g_{J_2} + \mu(t)^{*t}(h - \dot{y}^*))$, we get

$$b_0(x^*, y^*, u^*, v^*) \left[\int_a^b \lambda^*_{J_1}(t)^t g_{J_1}(t, x^*, u^*) dt \right]$$

$$> b_1(x^*, y^*, u^*, v^*) \int_a^b [\lambda^*_{J_2}(t)^t g_{J_2}(t, y^*, v^*) + \mu^*(t)^t (h(t, y^*, v^*) - \dot{y}^*))] dt \geqq 0. \tag{48}$$

On the other hand, $\lambda^*_{J_1}(t) \geqq 0$, $g_{J_1}(t, x^*, u^*) \leqq 0$, hence with $b_0(x^*, y^*, u^*, v^*) \geqq 0$

$$b_0(x^*, y^*, u^*, v^*) \int_a^b \lambda^*_{J_1}(t)^t g_{J_1}(t, x^*, u^*) dt \leqq 0,$$

leads to a contradiction to (48). □

5 Conclusion

In this paper, we have introduced the classes of (B, ρ)-type I and generalized (B, ρ)-type I, and have used these different classes of functions to derive various sufficient optimality conditions and mixed type duality results for multiobjective control problems. Results for multiobjective variational problems can be obtained in similar lines.

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